UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education Advanced Level

FURTHER MATHEMATICS **9231/11**

Paper 1 **May/June 2012**

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF10)

* 2 6 1 3 2 7 7 6 9 3 *

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

UNIVERSITY of CAMBRIDGE
International Examinations

1 The roots of the cubic equation $x^3 - 7x^2 + 2x - 3 = 0$ are α, β, γ. Find the values of

(i) $\alpha^2 + \beta^2 + \gamma^2$,

(ii) $\alpha^3 + \beta^3 + \gamma^3$.

2 Prove, by mathematical induction, that, for integers $n \geqslant 2$,

$$4^n > 2^n + 3^n.$$ [5]

3 Given that $f(r) = \dfrac{1}{(r+1)(r+2)}$, show that

$$f(r-1) - f(r) = \frac{2}{r(r+1)(r+2)}.$$ [2]

Hence find $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$. [3]

Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$. [1]

4 The curve C has polar equation $r = 2 + 2\cos\theta$, for $0 \leqslant \theta \leqslant \pi$. Sketch the graph of C. [2]

Find the area of the region R enclosed by C and the initial line. [4]

The half-line $\theta = \frac{1}{5}\pi$ divides R into two parts. Find the area of each part, correct to 3 decimal places. [3]

5 A matrix $\mathbf{A}$ has eigenvalues -1, 1 and 2, with corresponding eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix},$$

respectively. Find $\mathbf{A}$. [9]

6 Write down the values of θ, in the interval $0 \leqslant \theta < 2\pi$, for which $\cos\theta + \mathrm{i}\sin\theta$ is a fifth root of unity. [2]

By writing the equation $(z+1)^5 = z^5$ in the form

$$\left(\frac{z+1}{z}\right)^5 = 1,$$

show that its roots are

$$-\frac{1}{2}\left\{1 + \mathrm{i}\cot\left(\frac{k\pi}{5}\right)\right\}, \quad k = 1, 2, 3, 4.$$ [7]

7 The linear transformations $\mathrm{T}_1 : \mathbb{R}^4 \to \mathbb{R}^4$ and $\mathrm{T}_2 : \mathbb{R}^4 \to \mathbb{R}^4$ are represented by the matrices

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 1 & 1 & 4 \\ 2 & 1 & 4 & 11 \\ 3 & 4 & 1 & 9 \\ 4 & -3 & 18 & 37 \end{pmatrix} \quad \text{and} \quad \mathbf{M}_2 = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 2 & 3 & 0 & 1 \\ 3 & 4 & 1 & 0 \\ 4 & 5 & 2 & 0 \end{pmatrix},$$

respectively. The null space of T_1 is denoted by K_1 and the null space of T_2 is denoted by K_2. Show that the dimension of K_1 is 2 and that the dimension of K_2 is 1. [5]

Find the basis of K_1 which has the form $\left\{ \begin{pmatrix} p \\ q \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} r \\ s \\ 0 \\ 1 \end{pmatrix} \right\}$ and show that K_2 is a subspace of K_1. [5]

8 Find the particular solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 10\mathrm{e}^{-2x},$$

given that $y = 5$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$. [11]

9 The curve C has equation

$$y = \frac{2x^2 + 2x + 3}{x^2 + 2}.$$

Show that, for all x, $1 \leqslant y \leqslant \frac{5}{2}$. [4]

Find the coordinates of the turning points on C. [3]

Find the equation of the asymptote of C. [2]

Sketch the graph of C, stating the coordinates of any intersections with the y-axis and the asymptote. [2]

10 The curve C has equation

$$y = 2\left(\frac{x}{3}\right)^{\frac{3}{2}},$$

where $0 \leqslant x \leqslant 3$. Show that the arc length of C is $2(2\sqrt{2} - 1)$. [4]

Find the coordinates of the centroid of the region enclosed by C, the x-axis and the line $x = 3$. [7]

11 Answer only **one** of the following two alternatives.

EITHER

Show that

$$\int_0^{\pi} e^x \sin x \, dx = \frac{1 + e^{\pi}}{2}. \qquad [4]$$

Given that

$$I_n = \int_0^{\pi} e^x \sin^n x \, dx,$$

show that, for $n \geqslant 2$,

$$I_n = n(n-1) \int_0^{\pi} e^x \cos^2 x \sin^{n-2} x \, dx - nI_n,$$

and deduce that

$$(n^2 + 1)I_n = n(n-1)I_{n-2}. \qquad [6]$$

A curve has equation $y = e^x \sin^5 x$. Find, in an exact form, the mean value of y over the interval $0 \leqslant x \leqslant \pi$. [4]

OR

The position vectors of the points A, B, C, D are

$$2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}, \quad -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}, \quad \mathbf{i} + 4\mathbf{j} + \mathbf{k}, \quad \mathbf{i} + 5\mathbf{j} + m\mathbf{k},$$

respectively, where m is an integer. It is given that the shortest distance between the line through A and B and the line through C and D is 3. Show that the only possible value of m is 2. [7]

Find the shortest distance of D from the line through A and C. [3]

Show that the acute angle between the planes ACD and BCD is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$. [4]